INVITATION TO INTER-UNIVERSAL TEICHMÜLLER THEORY (2+2 HOUR VERSION)

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"Travel and Lectures"

§1. Hodge-Arakelov-theoretic Motivation

§2. Teichmüller-theoretic Deformations

§3. The Log-theta-lattice

§4. Inter-universality and Anabelian Geometry

§1. **Hodge-Arakelov-theoretic Motivation**

We begin with a concrete argument. Suppose that $h \in \mathbb{R}_{\geq 0}$ is a **quantity** that we wish to **bound from above**, and that, for some integer $N \geq 2$, we know that the equality

$$N \cdot h \ = \ h$$

holds. Then an easy algebraic manipulation shows that

$$(N-1) \cdot h \ = \ 0, \qquad \text{i.e.,} \qquad h \ = \ 0.$$

An easy variant of this argument involves a "relatively small" constant $C \in \mathbb{R}$ for which the inequality

$$N \cdot h \ \leq \ h + C$$

holds; this inequality implies that

$$(N-1) \cdot h \ \leq \ C, \qquad \text{i.e.,} \qquad h \ \leq \ \tfrac{1}{N-1} \cdot C$$

— that is to say, that h can indeed be **bounded from above**, as desired.

Next, we consider **elliptic curves**. For l a prime number, the module of l-torsion points associated to a **Tate curve** $E \stackrel{\text{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ (over, say, a p-adic field or $\mathbb{C}$) fits into a natural exact sequence:

$$0 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 0$$

That is to say, one has **canonical** objects as follows:

a "**multiplicative subspace**" $\boldsymbol{\mu}_l \subseteq E[l]$ and "**generators**" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$

In the following, we fix an **elliptic curve** E over a **number field** F and a **prime number** $l \geq 5$. Also, we suppose that E has stable reduction at all finite places of F. Then, in general, $E[l]$ does **not** admit

a **global** "**multiplicative subspace**" and "**generators**"

that coincide with the above canonical "multiplicative subspace" and "generators" at all finite places where E has **bad multiplicative reduction**!

Nevertheless, let us **suppose** (!!) that such global objects do in fact exist. Then, if we denote by $E \to E^*$ the isogeny obtained by forming the **quotient** of E by the "**global multiplicative subspace**", then, at each finite prime of bad multiplicative reduction, the respective q-parameters satisfy the following relation:

$$q_E^l \ = \ q_{E^*}$$

If we write $\log(q_E)$, $\log(q_{E^*})$ for the **arithmetic degrees** $\in \mathbb{R}$ determined by these q-parameters, then the above relation takes on the following form:

$$l \cdot \log(q_E) \ = \ \log(q_{E^*}) \ \in \mathbb{R}$$

On the other hand, if we denote the respective **heights** of the elliptic curves by $\mathrm{ht}_E, \mathrm{ht}_{E^*} \in \mathbb{R}$, then we may conclude that

$$\mathrm{ht}_{(-)} \approx \tfrac{1}{6} \cdot \log(q_{(-)})$$

(where "$\approx$" means "up to a discrepancy bounded by a constant"). Moreover, by the famous **computation concerning differentials** due to Faltings (1983), one knows that:

$$\mathrm{ht}_{E^*} \ \lesssim \ \mathrm{ht}_E + \log(l)$$

Thus, just as in the argument given prior to the present discussion of elliptic curves, we conclude that

$$l \cdot \mathrm{ht}_E \ \lesssim \ \mathrm{ht}_E + \log(l), \qquad \text{i.e.,} \qquad \mathrm{ht}_E \ \lesssim \ \tfrac{1}{l-1} \cdot \log(l) \ \lesssim \ \text{constant}$$

— that is to say, that the height ht_E of the elliptic curve E can be **bounded from above**, and hence (under suitable hypotheses) that there are only **finitely many** isomorphism classes of elliptic curves E that admit a "global multiplicative subspace".

Ultimately, we would like to **generalize** the above argument to the case of **general elliptic curves** for which "global multiplicative subspaces", etc. do not necessarily exist. But, before doing so, we would like to consider an approach that is slightly different from the above argument, still under the (**unrealistic**!) assumption that such global objects do indeed exist.

To this end, it is necessary to "review" the **Fundamental Theorem** of **Hodge-Arakelov Theory** under this assumption that such global objects do indeed exist.

The **Fundamental Theorem** of **Hodge-Arakelov Theory** may be formulated as follows:

$$\Gamma(E^{\dagger}, \mathcal{L})^{<l} \quad \xrightarrow{\sim} \quad \bigoplus_{j=-l^*}^{l^*} \left(\underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F \right) \otimes F$$

— where

· $E^{\dagger} \to E$ is the "**universal vectorial extension**" of E;

· "$< l$" is the "relative degree" w.r.t. this extension; $l^* \stackrel{\mathrm{def}}{=} (l-1)/2$;

· $\mathcal{L}$ is a line bundle that arises from a (nontrivial) 2-torsion point;

· "q" is the q-parameter at bad places of F; $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} q^{1/2l}$;

· the **LHS** admits a **Hodge filtration** F^{-i} s.t. F^{-i}/F^{-i+1} is (roughly)

$$\xrightarrow{\sim} \ \omega_E^{\otimes(-i)} \qquad (i = 0, 1, \ldots, l-1;\ \omega_E = \text{cotang. bun. at the origin});$$

· the **RHS** admits a natural **Galois action** compatible with "$\bigoplus$".

This isom. is, *a priori*, only defined/F, but is in fact (essentially) **compatible** with the natural **integral structures**/**metrics** at all places of F.

A similar isom. may be considered over the **moduli stack** of **ell. curves**. The proof of such an isom. is based on a **computation**, which shows that the **degrees** $[-]$ of the vector bundles on either side of the isom. **coincide**:

$$\tfrac{1}{l} \cdot \mathrm{LHS} \ \approx \ -\tfrac{1}{l} \cdot \textstyle\sum_{i=0}^{l-1} \ i \cdot [\omega_E] \ \approx \ -\tfrac{l}{2} \cdot [\omega_E]$$

$$\tfrac{1}{l} \cdot \mathrm{RHS} \ \approx \ -\tfrac{1}{l^2} \cdot \textstyle\sum_{j=1}^{l^*} \ j^2 \cdot [\log(q)] \ \approx \ -\tfrac{l}{24} \cdot [\log(q)] \ = \ -\tfrac{l}{2} \cdot [\omega_E]$$

On the other hand, returning to the situation over **number fields**, since F^i is **not compatible** with the above **direct sum decomposition**, it follows that, by projecting to the factors of this direct sum decomposition, one may construct a sort of relative of the so-called "**arithmetic Kodaira-Spencer morphism**", i.e., for (most) j, a (nonzero) morphism of line bundles:

$$(\mathcal{O}_F \ \approx) \ \ F^0 \ \hookrightarrow \ \underline{\underline{q}}^{j^2} \cdot \mathcal{O}_F.$$

Since, moreover, $\deg_{\text{arith}}(F^0) \approx 0$, it follows that, if we denote the height determined by the **logarithmic differentials** $\Omega^{\log}_{\mathcal{M}}|_E$ associated to the moduli stack of elliptic curves by $\text{ht}_E \stackrel{\text{def}}{=} 2 \cdot \deg_{\text{arith}}(\omega_E) = \deg_{\text{arith}}(\Omega^{\log}_{\mathcal{M}}|_E)$, then we obtain an **inequality** (!) as follows:

$$\frac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \ = \ \text{ht}_E \ < \ \text{constant}$$

In fact, of course, since the **global** **mult. subspace** and **generators** which play an essential role in the above argument do **not**, in general, **exist**, this argument cannot be applied immediately in its present form.

This state of affairs motivates the following approach, which may appear somewhat **far-fetched** at first glance! Suppose that the assignment

$$\left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$

somehow determines an **automorphism** of the **number field** F! Such an "automorphism" necessarily **preserves** **degrees of arith. line bundles**. Thus, since the absolute value of the degree of the **RHS** of the above assignment is "small" by comparison to the absolute value of the (**average**!) degree of the **LHS**, we thus conclude that a similar **inequality** (!) holds:

$$\frac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \ = \ \text{ht}_E \ < \ \text{constant}$$

Of course, such an automorphism of a NF does **not** in fact **exist**!! On the other hand, what happens if we regard the "$\{\underline{\underline{q}}^{j^2}\}$" on the LHS and the "$\underline{\underline{q}}$" on the RHS as belonging to **distinct** copies of "**conventional ring/scheme theory**" = "**arithmetic holomorphic structures**", and we think of the assignment under consideration

$$\left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*} \quad \mapsto \quad \underline{\underline{q}}$$

— i.e., which may be regarded as a sort of "**tautological solution**" to the

"**obstruction to applying HA theory to diophantine geometry**"

— as a sort of **quasiconformal map** between Riemann surfaces equipped with **distinct holomorphic structures**?

Remark: The method of first preparing a "**supplementary stage**" on which a "**taut. solution**" holds, then proceeding to **compare** this supplementary stage with the "original stage" (i.e., isomorphic? almost isom'ic?) is a **standard tool** that is frequently applied in arithmetic geometry, or, indeed, in mathematics in general. Put another way, this method consists of allowing ourselves first to acquire the "taut. solution" **for free** and then proceeding to calculate the **costs** incurred as a result of this acquisition.

Classical examples of "taut. solutions": The mechanism (which dates back to ancient civilizations!) whereby one applies **borrowed assets** to conduct business, which leads to the creation of **new assets** that render it possible to reimburse the borrowed assets **with interest**. The introduction of **indeterminates** and **algebraic manipulations** into a situation in which only addition, subtraction, multiplication, and division of rational numbers are known. The introduction of **abstract "fields"** and **Galois groups** into a situation in which only explicit formulas via radicals for roots of equations of degree 2 to 4 are known.

That is to say, (returning to the above discussion of elliptic curves over number fields) this approach allows us to realize the assignment under consideration, albeit at the cost of **partially dismantling** conventional ring/sch. theory. On the other hand, this approach requires us

to compute just how much of a distortion occurs

as a result of dismantling = **deforming** conventional ring/scheme theory. This **vast computation** is the **content of IUTeich**.

In conclusion, at a concrete level, the "distortion" that occurs at the portion labeled by the index j is (roughly)

$$\leq \ j \cdot (\text{log-diff}_F + \text{log-cond}_E).$$

In particular, by the **exact same** computation (i.e., of the "leading term" of the **average** over j) as the computation discussed above in the case of the moduli stack of elliptic curves, we obtain the **inequality** of the so-called **Szpiro Conjecture** ($\Longleftrightarrow$ **ABC Conjecture**):

$$\frac{1}{6} \cdot \deg_{\text{arith}}(\log(q)) \ = \ \text{ht}_E \ \leq \ (1+\epsilon)(\text{log-diff}_F + \text{log-cond}_E) \ + \ \text{constant}$$

Remark: The equality "$N \cdot h \ = \ h$" considered above corresponds to

the "**tautological solution**" $\iff$ "$N \cdot h^{\mathrm{LHS}} = h^{\mathrm{RHS}}$"

— i.e., where one **distinguishes** the **LHS** and **RHS** "h's". Once one allows for the ensuing **distortions**, one may then **identify** these two "h's" and conclude that "$N \cdot h \ \leq \ h + C$" $\iff$ the Szpiro Conjecture inequality.

§2. Teichmüller-theoretic Deformations

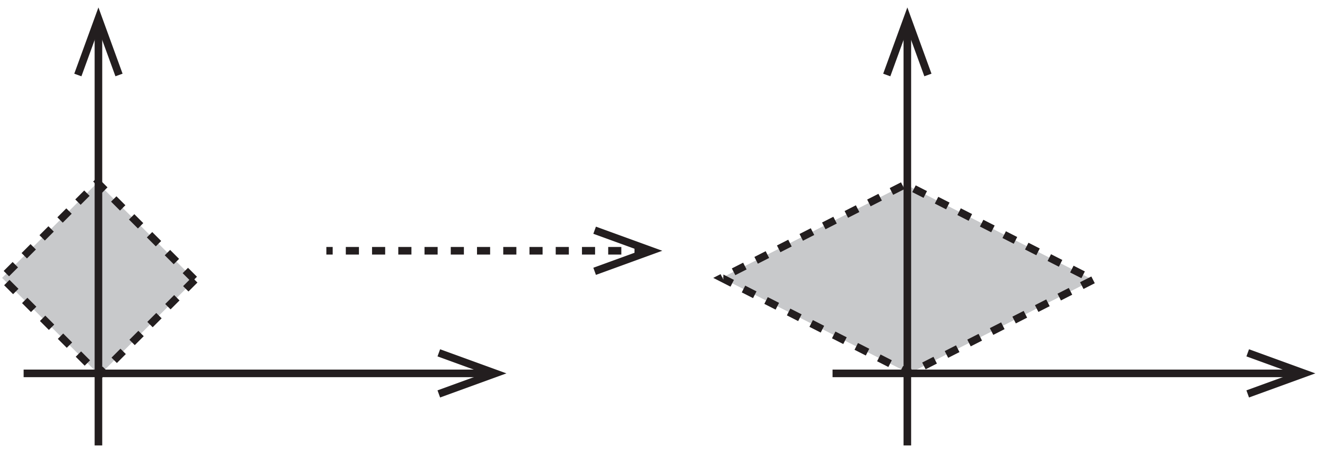
Classical Teichmüller theory over $\mathbb{C}$:

Relative to a canonical coordinate $z = x + iy$ (associated to a square differential) on the Riemann surface, **Teich. deformations** are given by

$$z \ \mapsto \ \zeta \ = \ \xi + i\eta = \ Kx + iy$$

— where $1 < K < \infty$ is the **dilation** factor.

Key point: **one** holomorphic dim., but **two** underlying real dims., of which **one** is **dilated/deformed**, while the **other** is left **fixed/undeformed**!



p-adic Teichmüller theory:

· **p-adic canonical liftings** of a hyperbolic curve in positive characteristic equipped with a nilpotent indigenous bundle

· **canonical Frobenius liftings** over the ordinary locus of the moduli stack of curves, as well as over the tautological curve — cf. the metric on the **Poincaré** upper half-plane, **Weil-Petersson metric** in the theory/$\mathbb{C}$.

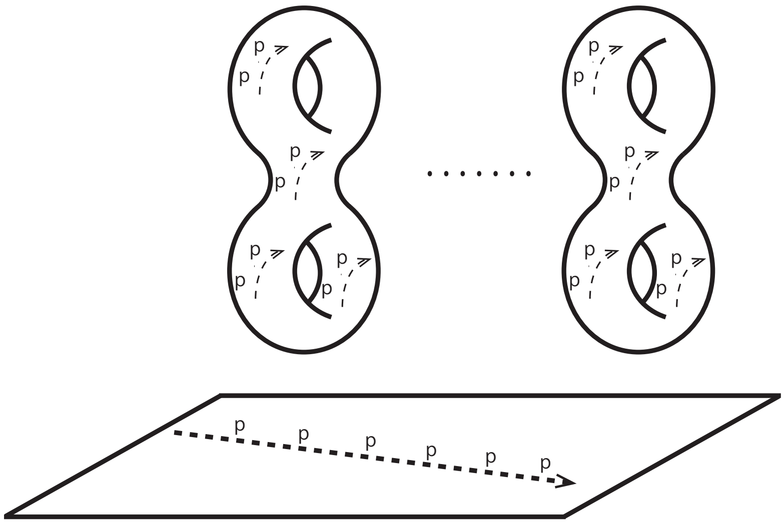
Analogy betw. IUTeich and pTeich (cf. **univ. ell. curve**/$\mathbb{P}^1 \setminus \{0, 1, \infty\}$!):

conventional scheme theory/$\mathbb{Z}$ $\longleftrightarrow$ scheme theory/$\mathbb{F}_p$

number field (+ fin. many places) $\longleftrightarrow$ hyperbolic curve in pos. char.

once-punctured elliptic curve/NF $\longleftrightarrow$ nilpotent indigenous bundle

log-Θ-lattice $\longleftrightarrow$ **p-adic canonical lifting** + **canonical Frob. lifting**



The arithmetic case: addition and multiplication, cohom. dim.:

Regard the **ring structure** of rings such as $\mathbb{Z}$ as a

one-dimensional "arithmetic holomorphic structure"!

— which has **two underlying combinatorial dimensions**!

"addition"	and	"multiplication"
$(\mathbb{Z}, +)$	$\curvearrowleft$	$(\mathbb{Z}, \times)$
one combinatorial dim.		one combinatorial dim.

— cf. the **two cohomological dims.** of the absolute Galois group of

· a (totally imaginary) number field $F/\mathbb{Q} < \infty$,

· a p-adic local field $k/\mathbb{Q}_p < \infty$,

(Note: the pro-l-related portion of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is $\approx \mathbb{Z}_l \rtimes \mathbb{Z}_l^{\times}$),

as well as the **two underlying real dims.** of

· $\mathbb{C}^{\times}$.

<u>Units and value groups</u>:

In the case of a p-adic local field $k/\mathbb{Q}_p < \infty$, one may also think of these **<u>two underlying</u>** **<u>combinatorial dimensions</u>** as follows:

$$\begin{array}{ccccc} \mathcal{O}_k^\times & \subseteq & k^\times & \twoheadrightarrow & k^\times/\mathcal{O}_k^\times \ (\cong \mathbb{Z}) \\ \text{one combinatorial dim.} & & & & \text{one combinatorial dim.} \end{array}$$

— cf. the direct product decomp. in the complex case: $\mathbb{C}^\times = \mathbb{S}^1 \times \mathbb{R}_{>0}$.

In IUTeich, we shall **<u>deform the holom. str. of the number field</u>** by

<u>dilating the value groups</u> via the **<u>theta function</u>**, while

leaving the **<u>units undilated</u>**!

§3. <u>The Log-theta-lattice</u>

<u>Noncommutative (!) 2-dim. diagram of Hodge theaters "•"</u>:

2 dims. of the diagram $\longleftrightarrow$ **<u>2 comb. dims.</u>** of a p-adic local field!

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

<u>Analogy between IUTeich and pTeich</u> (cf. also the **<u>Witt vectors</u>**!):

$\bullet$ = a copy of scheme theory/$\mathbb{Z}$ $\longleftrightarrow$ a copy of scheme theory/$\mathbb{F}_p$

$\uparrow$ = $\mathfrak{log}$-link = **<u>gluing</u>** betw. two copies $\longleftrightarrow$ the Frob. mor. in pos. char.

$\longrightarrow$ = Θ-link = **<u>gluing</u>** betw. two copies $\longleftrightarrow$ $\left(p^n/p^{n+1} \rightsquigarrow p^{n+1}/p^{n+2}\right)$

[$\Theta^{\pm\mathrm{ell}}$NF-]Hodge theaters:

A "[$\Theta^{\pm\mathrm{ell}}$NF-]Hodge theater" is a model of the **conventional scheme-theoretic arithmetic geometry** surrounding an elliptic curve E over a number field F. At a more concrete level, it is a complicated **system** of

abstract monoids and **Galois groups/arith. fund. gps.**

that arise naturally from E/F and its various localizations.

The **principle** that underlies this system: the system serves as
a **bookkeeping apparatus** for the ***l*-tors. points** that allows one
to simulate a **global** **multiplicative subspace** + **generators** (cf. §1)!

$$\rightsquigarrow \ \mathbb{F}_l^{\circledast}\text{-, } \mathbb{F}_l^{\rtimes\pm}\text{-symmetries}$$

$$(\text{where} \quad \mathbb{F}_l^{\circledast} \stackrel{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}, \quad \mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\})$$

$$\overset{\{\pm 1\}}{\curvearrowright} \begin{pmatrix} -l^{*} < \dots < -1 < 0 \\ < 1 < \dots < l^{*} \end{pmatrix} \Rightarrow \begin{bmatrix} 1 < \dots \\ < l^{*} \end{bmatrix} \Leftarrow \begin{pmatrix} 1 < \dots \\ < l^{*} \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} \qquad\qquad \begin{array}{ccc} \circledast & \rightarrow & \circledast \\ \uparrow & \overset{\mathbb{F}_l^{\circledast}}{\curvearrowright} & \downarrow \\ \circledast & \leftarrow & \circledast \end{array}$$

...cf. **ord. monodromy**! ...cf. **s/sing.**!

Hint that underlies the construction of this apparatus: structure of ***p*-Hecke correspondence** special fiber; **global multiplicative subspace** on the moduli stack of elliptic curves over $\mathbb{Q}_p$, as in ***p*-adic Hodge theory**

(p-adic Tate module) $\otimes$ (p-adic ring of fns.)

... $\rightsquigarrow$ "**combinatorial rearrangement**" of **basepoints** by means
of a **mysterious** '$\otimes$'!

$\longleftrightarrow$ the **absolute anabelian geometry** applied in a Hodge theater!
= "**invariant**" w.r.t. **two distinct roles** played by 'ring of fns.'

$\mathfrak{log}$-Link:

At **nonarchimedean** v of the number field F, the **ring structures** on either side of the $\mathfrak{log}$-link are related by a **non-ring-homomorphism** (!)

$$\log_v : \mathcal{O}^{\times}_{\overline{k}} \ \to \ \overline{k}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\text{def}}{=} F_v$; $G_v \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$.

Key point: The $\mathfrak{log}$-link is **compatible** with the isomorphism

$$\Pi_v \ \stackrel{\sim}{\to} \ \Pi_v$$

between the arithmetic fundamental groups Π_v on either side of the $\mathfrak{log}$-link, relative to the **natural actions** via $\Pi_v \twoheadrightarrow G_v$. Moreover, if one allows v to vary, the $\mathfrak{log}$-link is also compatible with the action of the **global absolute Galois groups**. Finally, at **archimedean** v of F, one has an analogous theory.

Θ-Link:

At **bad nonarchimedean** v of the number field F, the **ring structures** on either side of the Θ-link are related by a **non-ring-homomorphism** (!)

$$\mathcal{O}^{\times}_{\overline{k}} \ \stackrel{\sim}{\to} \ \mathcal{O}^{\times}_{\overline{k}}; \qquad (\Theta|_{l\text{-tors}})^{\mathbb{N}} = \left\{ \underline{\underline{q}}^{j^2} \right\}^{\mathbb{N}}_{j=1,\dots,l^*} \ \mapsto \ \underline{\underline{q}}^{\mathbb{N}}$$

— where $\overline{k}$ is an algebraic closure of $k \stackrel{\text{def}}{=} F_v$; $G_v \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$.

Key point: The Θ-link is **compatible** with the isomorphism

$$G_v \ \stackrel{\sim}{\to} \ G_v$$

between the Galois groups G_v on either side of the Θ-link, relative to the **natural actions** on $\mathcal{O}^{\times}_{\overline{k}}$. At **good nonarchimedean/archimedean** v of F, one can give an analogous definition, by applying the **product formula**.

Remark: It is only possible to define the "**walls/barriers**" (i.e., from the point of view of the **ring structure** of conventional ring/scheme theory) constituted by the $\mathfrak{log}$-, Θ-links by working with

abstract monoids/...

— i.e., of the sort that appear in a Hodge theater!

Remark: By contrast, the objects that appear in the **étale-picture** (cf. the diagram below!) — i.e., the portion of the log-theta-lattice constituted by the

arithmetic fundamental groups/Galois groups

— have the power to

slip through these “walls”!

Various versions of “**Kummer theory**” — which allow us to relate the following two types of mathematical objects:

abstract monoids = **Frobenius-like** objects and
arith. fund. gps./Galois groups = **étale-like** objects

— play a very important role throughout IUTeich! Moreover, the transition

Frobenius-like $\rightsquigarrow$ **étale-like**

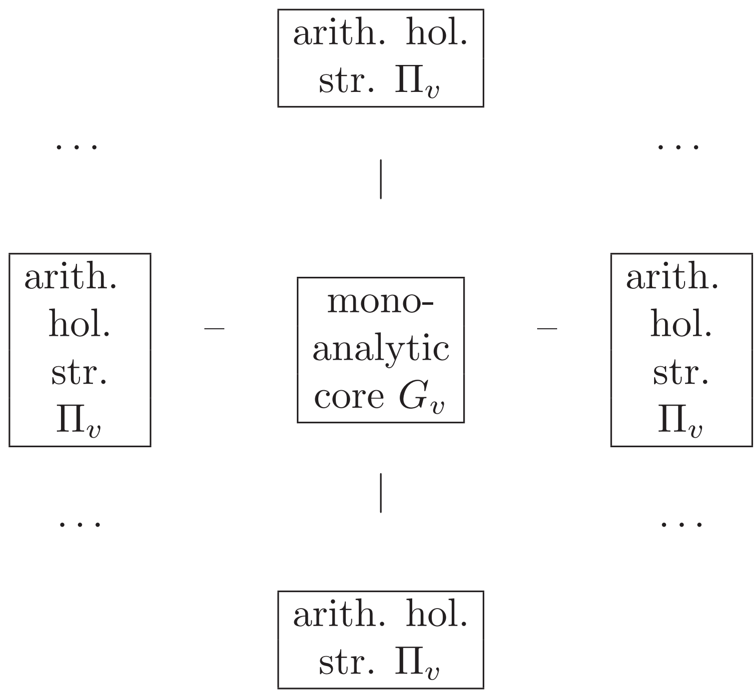
may be regarded as a **global analogue over number fields** of the computation — i.e., via “**cartesian coords.** $\rightsquigarrow$ **polar coords.**” — of the classical **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx \ = \ \sqrt{\pi} \qquad \Big(= \ \text{“weight } \tfrac{1}{2}\text{” } ! \ \notin \ \mathbb{Q} \cdot \pi^{\mathbb{Z}} \ \ni \ \zeta(n \in 2 \cdot \mathbb{Z}) \Big) \ !$$

Indeed, the coord. trans. $e^{-r^2} \rightsquigarrow u$ that appears in this computation

$$2 \cdot (\textstyle\int e^{-x^2} dx)^2 \ = \ 2 \cdot \int\int e^{-x^2-y^2} \ dx \ dy \ = \ \int\int e^{-r^2} \cdot 2r dr \ d\theta$$
$$= \ \textstyle\int\int d(e^{-r^2}) \ d\theta \ = \ \int\int du \ d\theta$$

is quite similar in form to the Θ-link!



Main objects to which **Kummer theory** is **applied** (cf. **LHS** of **Θ-link**!):

(a) **gp. of units** $\quad \mathcal{O}^{\times}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$ (nonarch. v)

(b) **values of theta fn.** $\quad \Theta|_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*}$ (bad nonarch. v)

(c) a sort of "**realification**" of the **number field** F ($\supseteq F^{\times}$)

Main focus of the theory is to protect the **cyclotomes** ($\cong \widehat{\mathbb{Z}}(1)$) contained in the monoids where (b), (c) appear from the **indeterminacy** "$\curvearrowleft \widehat{\mathbb{Z}}^{\times}$", i.e., **cyclotomic rigidity**!

Case of (b): **theory of étale theta fn.** $\Longrightarrow$ cyclo. rig.
Case of (c): **elem. alg. no. theory** $\Longrightarrow$ cyclo. rig.

The Kummer theory of (b), (c) is well-suited to the resp. portions of a Hodge theater where the **symmetries** act (cf. the chart below)!

This state of affairs closely resembles the (well-known) elementary theory of the "**functions**" associated to the various **symmetries** of the classical upper half-plane $\mathfrak{H}$ (cf. the chart below)!

	The classical upper half-plane $\mathfrak{H}$	**$\Theta^{\pm\text{ell}}$NF-Hodge theaters in IUTch**
(Cuspidal) add. symm.	$z \mapsto z + a$, $z \mapsto -\overline{z} + a \quad (a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$-**symmetry**
"Functions" assoc. to add. symm.	$q \stackrel{\text{def}}{=} e^{2\pi i z}$	$\Theta\vert_{l\text{-tors}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*}$
(Nodal/toral) mult. symm.	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)} \quad (t \in \mathbb{R})$	$\mathbb{F}_l^{*}$-**symmetry**
"Functions" assoc. to mult. symm.	$w \stackrel{\text{def}}{=} \frac{z-i}{z+i}$	representation of elts. of **no. fld.** F via **Belyi maps**

In fact, this portion of IUTeich closely resembles, in many respects (cf. the chart below!), **Jacobi's identity**

$$\theta(t) \;=\; t^{-1/2} \cdot \theta(1/t)$$

— which may be thought of as a sort of **function-theoretic** version of the **Gaussian integral** that appeared in the discussion above — concerning the classical **theta function** on the upper half-plane

$$\theta(t) \;=\; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

IUTeich	**Theory of Jacobi's identity**
rigidity properties of étale theta fn.	**invariance** of **Gaussian distrib.** w.r.t. **Fourier transform**
the **indeterminacy** $\mathcal{O}^{\times}_{\overline{k}} \curvearrowleft \widehat{\mathbb{Z}}^{\times}$	**unit factor** in Fourier transform $\int (-) \cdot e^{it}$, $t \in \mathbb{R}$
proof of rig. properties via **quad'icity** of theta gp. $[-,-]$	proof of Fourier invariance via **quad'icity** of exp. of Gauss. dist.
$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*}$	**Gaussian** expansion of theta fn.
Abs. anab. geom. applied to **rotation** of $\boxplus$, $\boxtimes$ via $\mathfrak{log}$-link	**Analytic continuation** $\infty \rightsquigarrow 0$, the **rotation** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \iff t \mapsto \frac{1}{t}$
Local/global functoriality of abs. anab. algorithms, **Belyi cuspidalization**	**Analytic continuation** to the **entire upper half-plane**

One way to interpret **Jacobi's identity**

$$\theta(t) \;=\; t^{-1/2} \cdot \theta(1/t)$$

concerning the classical **theta function** on the upper half-plane

$$\theta(t) \;=\; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

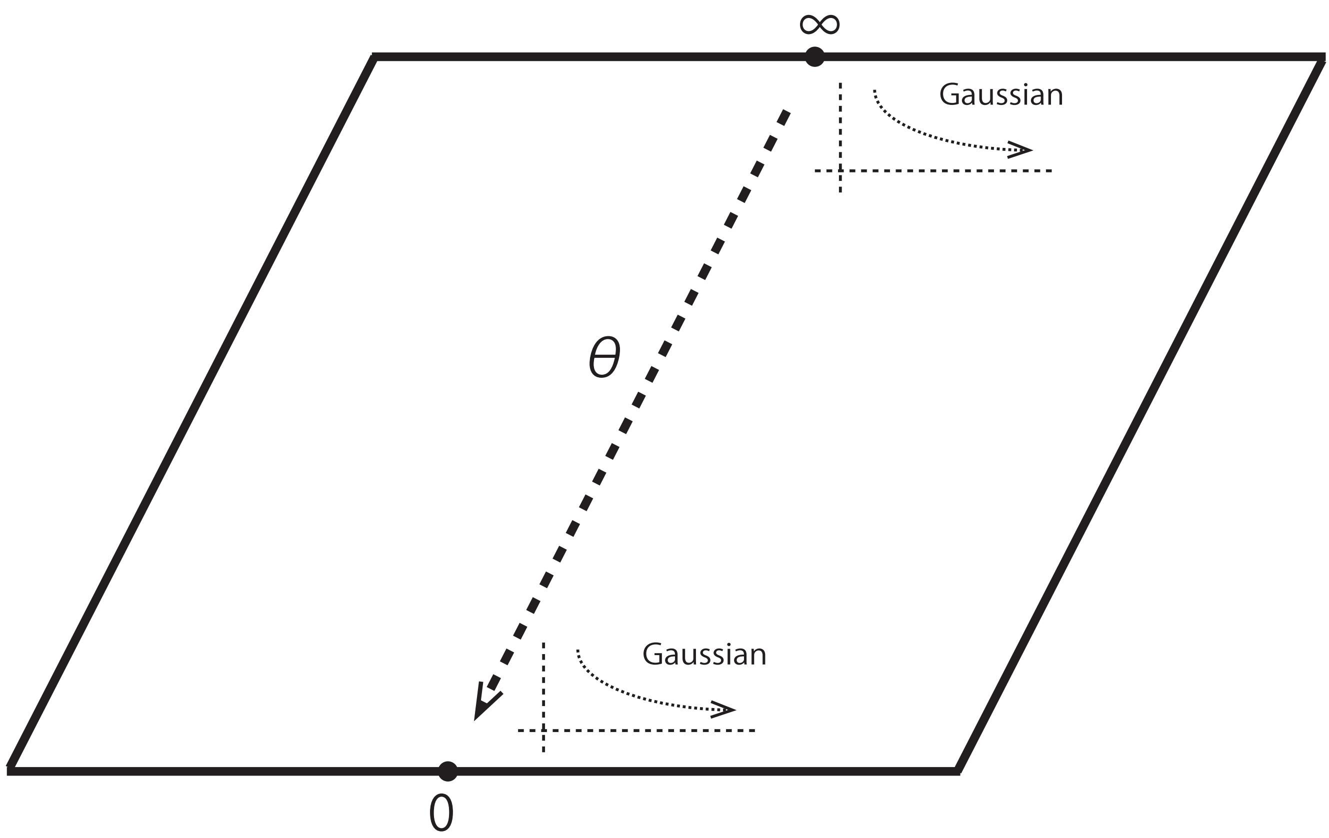
is as follows: the property of being representable, in a neighborhood of ∞, by a series in which each nonconstant summand is a **rapidly decreasing function** continues to hold — somewhat **mysteriously**! — even after **analytic continuation**

$$\infty \;\rightsquigarrow\; 0!$$

This phenomenon corresponds to the **global** realization of the expression

$$\text{“}\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*}\text{”}$$

discussed in §1!



§4. **Inter-universality and Anabelian Geometry**

Note that the $\mathfrak{log}$-, Θ-links are **not compatible** with the **ring structures**

$$\log_v : \mathcal{O}_{\overline{k}}^{\times} \ \to \ \overline{k}, \qquad (\Theta|_{l\text{-tors}})^{\mathbb{N}} = \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*}^{\mathbb{N}} \ \mapsto \ \underline{\underline{q}}^{\mathbb{N}}$$

in their domains and codomains, hence are **not compatible**, in a quite **essential** way, with the **scheme-theoretic** "**basepoints**" and

Galois groups ($\subseteq \mathrm{Aut}_{\mathrm{field}}(\overline{k})$!!)

that arise from **ring homomorphisms**! That is to say, when one passes to the "opposite side" of the $\mathfrak{log}$-, Θ-links,

"Π_v" and "G_v"

only make sense in their capacity as **abstract topological groups** (cf. the outer autom. of absolute Galois groups induced by an autom. of a field)!

$\Longrightarrow$ As a consequence, in order to compute the relationship between the ring structures in the domain and codomain of the $\mathfrak{log}$-, Θ-links, it is necessary to apply **anabelian geometry**! At the level of previous papers by the author, we derive the following Main Theorem by applying the results and theory of

- Semi-graphs of Anabelioids
- The Geometry of Frobenioids I, II
- The Étale Theta Function ...
- Topics in Absolute Anab. Geo. III

concerning

absolute anabelian geometry and

various rigidity properties of the étale theta function.

Main Theorem: One can give an **explicit, algorithmic description**, up to mild indeterminacies, of the **LHS** of the **Θ-link** in terms of the "**alien**" **ring structure** on the **RHS** of the Θ-link.

Interpretation: Even under circumstances where one is only linked by a "**narrow pipe**" (i.e., such as an astronaut on a space vessel or miners working in an underground mine), it is possible to reconstruct and grasp the situation on the "**other side**" by making wise use of the **limited information** available.

Key points:

· the **coricity** (i.e., coric nature) of $G_v \curvearrowright \mathcal{O}^{\times}_{\overline{k}}$!

· various versions of "**Kummer theory**", which allow us to relate the following two types of mathematical objects (cf. the latter portion of §3):

abstract monoids = **Frobenius-like** objects and
arith. fund. gps./Galois groups = **étale-like** objects.

Here, we recall the analogy with the computation of the **Gaussian integral**:

definition of **$\mathfrak{log}$-, Θ-link**, log-theta-lattice $\longleftrightarrow$ **cartesian coords.**

algor. descr. via **abs. anab. geom.** $\longleftrightarrow$ **polar coords.**

crucial rigidity of **cyclotomes** ($\cong \widehat{\mathbb{Z}}(1)$) $\longleftrightarrow$ coord. trans. via $\mathbb{S}^1 \curvearrowright$

· the **$\mathfrak{log}$-link** plays an indispensable role in the context of realizing the action on the "**log-shell**" = "**container**"

$$\log(\mathcal{O}^{\times}_{\overline{k}}) \curvearrowleft \left\{ \underline{\underline{q}}^{j^2} \right\}_{j=1,\dots,l^*}$$

... but various technical difficulties arise as a consequence of the **noncommutativity** of the **log-theta-lattice**.

$\Longrightarrow$ in the subsequent "volume computation", one only obtains an **inequality** (i.e., not an equality)!

By performing a **volume computation**, as discussed in §1, concerning the **output** of the algorithms of the above Main Theorem, one obtains (cf. the **class field theory**, ***p*-adic Hodge theory**, **algebraic geom.** related to **abelian vars.**, etc. that appear in Faltings' pf. of the Mordell Conjecture!):

Corollary: The "**Szpiro Conjecture**" ($\Longleftrightarrow$ "**ABC Conjecture**").

$$\mathrm{ht}_E \leq (1+\epsilon)(\text{log-diff}_F + \text{log-cond}_E) + \text{constant}$$

Here, we recall the arguments of §1: "$N \cdot h^{\mathrm{LHS}} = h^{\mathrm{RHS}}$" ($\longleftrightarrow$ Θ-link!), "$N \cdot h \leq h + C$" ($\longleftrightarrow$ Main Theorem + volume computation)!

This portion of the theory resembles, in many respects, the theory surrounding **Jacobi's identity**, as discussed at the end of §3:

IUTeich	**Theory of Jacobi's identity**
Changes of **universe**, i.e., **labeling apparatus** for **sets**	Changes of **coordinates**, i.e., **labeling apparatus** for **points** of a **space**
computation of volume of **log-shell** $\log(-)$	computation via **polar coordinates** of **Gaussian integral** $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
Startling application to **diophantine geometry**	**Startling** improvement in **computational accuracy** of values of classical theta function

In fact, the "ϵ" that appeared in the above inequality admits an **upper bound** of the following form:

$$(\mathrm{ht}_E)^{-\frac{1}{2}} \cdot \log(\mathrm{ht}_E)$$

Here, the "$\frac{1}{2}$" is reminiscent of the Riemann hypothesis. Indeed, just as in the case of the **Riemann hypothesis**, this "$\frac{1}{2}$" may be thought of as a phenomenon of

"**weight** 1/2"

(where the "weight" may be thought of as the "s" of the Riemann zeta function $\zeta(s)$), i.e., a phenomenon that concerns not integral powers of π, but rather the **square root of** π:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

In fact, the computation of this "ϵ" involves **quadratic forms** of the sort that appear in the Gaussian integral and the theta function; computation of the minimal value of such quadratic forms, i.e., of the roots of such quadratic forms, then gives rise to "**square roots**", i.e., to the quantity "$(\mathrm{ht}_E)^{-\frac{1}{2}}$".

Relative to the analogy with the classical theory concerning hyperbolic curves over p-adic local fields and the geometry of Riemann surfaces over $\mathbb{C}$, the corresponding **inequalities** (which may be regarded as expressions of "**hyperbolicity**") concerning the "**volume of a holomorphic structure**" computed from **outside the holomorphic structure** are as follows:

- the degree $= (2g-2)(1-p) \leq 0$ of the

$$\text{"}\textbf{Hasse invariant} = \frac{1}{p} \cdot d(\textbf{Frob. lift.})\text{"}$$

  in ***p*Teich**,
- the **Gauss-Bonnet Theorem** for a hyperbolic Riemann surface S

$$0 \ > \ -\int_S (\text{Poincaré metric}) \ = \ 4\pi(1-g).$$

Finally, I wish to point out another (more **elementary**) example of the "**spirit of inter-universal geometry**" — i.e.,

> "of an approach that yields **nontrivial** results in '**combinatorial**' situations in which conventional scheme theory is not available, by performing constructions that are **motivated** by conventional **scheme theory** and thus allow one to **approximate** conventional scheme theory to a substantial extent"

— namely,

combinatorial anabelian geometry

$\rightsquigarrow$ various results concerning the **GT group**

to effect that the **GT group** satsfies **analogous properties** to $G_{\mathbb{Q}}$ — i.e., **without** necessarily showing that it is $\cong G_{\mathbb{Q}}$!